INERTIAL ACCELERATED ALGORITHMS FOR SOLVING SPLIT FEASIBILITY PROBLEMS

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Abstract. Inspired by the inertial proximal algorithms for finding zero of a maximal monotone operator, we propose two inertial accelerated algorithms to solve the split feasibility problem. One is an inertial relaxed-CQ algorithm constructed by applying inertial technique to relaxed-CQ algorithm. The other one is a modified inertial relaxed-CQ algorithm which combines the KM method with the inertial relaxed-CQ algorithm. We prove their asymptotical convergence under certain suitable conditions. Numerical results are reported to show the effectiveness of the proposed algorithms.

1. Introduction. Given two nonempty convex closed sets, C and Q of real Hilbert spaces $H^1$ and $H^2$, respectively, and given a bounded linear operator $A : H^1 \rightarrow H^2$, consider the so-called split feasibility problem (SFP): Finding a point $x$ satisfying

$$x \in C, \ Ax \in Q.$$  (1.1)

This problem was originally introduced in [9], and has found applications in many fields such as image reconstruction, signal processing, and radiation therapy. A number of projection methods have been developed for solving SFP, see [6, 10, 23, 25]. Denote by $P_C$ the orthogonal projection onto C: $P_C(x) = \arg \min_{y \in C} \| x - y \|$, over all $x \in C$. In [5], Byrne introduced the so-called CQ algorithm, taking an initial point $x^0$ arbitrarily, and defining the iterative step as:

$$x^{k+1} = P_C[(I - \gamma A^T(I - P_Q)A)(x^k)],$$  (1.2)

where $0 < \gamma < 2/\rho(A^TA)$ and $\rho(A^TA)$ is the spectral radius of $A^TA$. The KM algorithm was proposed initially for solving fixed point problem [11], Byrne [6] first applied KM iterations to CQ algorithms for solving the SFP. Subsequently,
Zhao [26] applied KM iterations to perturbed CQ algorithms, Dang and Gao [14] combined the KM iterative method with the modified CQ algorithm to construct a KM-CQ-Like algorithm for solving the SFP. The implementation of the algorithms mentioned above is under the condition that the orthogonal projections onto $C$ and $Q$ are easily calculated. However, in most cases, it is impossible or needs too much work to compute the exact orthogonal projection. An inexact technique therefore plays an important role in designing efficiently and easily implemented algorithms for solving SFPs (see [15, 16]). An extension of the CQ algorithm that incorporates inexact techniques has been proposed by Qu and Xiu [22]. In [24], by using the relaxed projection technique, Yang developed a relaxed CQ algorithm for solving the SFP, where he used two halfspaces $C_k$ and $Q_k$ in place of $C$ and $Q$, respectively, at the $k$–th iteration such that the orthogonal projections onto $C_k$ and $Q_k$ are easily executed.

The problem of finding zero of a maximal monotone operator $G$ on a real Hilbert space $H$ is

$$\text{finding } x \in H \text{ such that } 0 \in Gx.$$ 

One of the fundamental approaches for solving this problem is the proximal method, which generates the next iteration $x^{k+1}$ by solving the subproblem

$$0 \in \lambda_k G(x) + (x - x^k),$$

where $x^k$ is the current iteration and $\lambda_k$ is a regularization parameter. In 2001, Attouch and Alvarez in [2] applied the inertial technique to the above algorithm (1.3) to present an inertial proximal method for solving the problem of finding zero of a maximal monotone operator. It works as follows. Given $x^{k-1}, x^k \in H$ and two parameters $\theta_k \in [0, 1), \lambda_k > 0$, find $x^{k+1} \in H$ such that

$$0 \in \lambda_k G(x^{k+1}) + x^{k+1} - x^k - \theta_k (x^k - x^{k-1}). \quad (1.4)$$

Here, the inertia is induced by the term $\theta_k (x^k - x^{k-1})$.

It is well known that the proximal iteration (1.3) may be interpreted as an implicit one-step discretization method for the evolution differential inclusion

$$0 \in \frac{dx}{dt} (t) + G(x(t)) \text{ a.e. } t \geq 0. \quad (1.5)$$

While the inspiration for (1.4) comes from the implicit discretization of the differential system of the second-order in time, namely

$$0 \in \frac{d^2 x}{dt^2} (t) + \rho \frac{dx}{dt} (t) + G(x(t)) \text{ a.e. } t \geq 0. \quad (1.6)$$

where $\rho > 0$ is a damping or a friction parameter. It gave rise to various numerical methods (for monotone inclusions and fixed problems) related to the inertial technology (first introduced in [2]), which, like (1.4), achieve nice convergence properties [1-3, 19, 20] by incorporating second order information.

Both the split feasibility problem and the problem of finding zero of a maximal monotone operator can be converted to the fixed point problem. Hence, inspired by the inertial proximal point algorithm for finding zeros of a maximal monotone operator, in this paper, we apply the inertial technique to the relaxed CQ method [24] to propose inertial relaxed-CQ algorithms for the SFP. Under certain suitable conditions. Asymptotical convergence of the algorithms is proved. Numerical experiments are reported to show the effectiveness of the algorithms.

The paper is organized as follows. In Section 2, we recall some preliminaries. In Section 3, we present an inertial relaxed-CQ algorithm and show its convergence. In
Section 4, a modified inertial relaxed-CQ algorithm is presented and its convergence is also proved. In Section 5, numerical experiments are given.

2. Preliminaries. Throughout the rest of the paper, \( I \) denotes the identity operator, \( \text{Fix}(T) \) denotes the set of the fixed points of an operator \( T \) i.e., \( \text{Fix}(T) := \{ x \mid x = T(x) \} \).

Recall that an operator \( T \) is called nonexpansive if
\[
\|T(x) - T(y)\| \leq \|x - y\|,
\]
firmly nonexpansive if
\[
\|T(x) - T(y)\|^2 \leq \langle x - y, T(x) - T(y) \rangle.
\]
It is well known that the orthogonal projection operator \( P_C \), for any \( x, y \), is characterized by the inequalities
\[
\langle x - P_C(x), c - P_C(x) \rangle \leq 0, \quad c \in C
\]
and
\[
\langle P_C(y) - P_C(x), y - x \rangle \geq \|P_C(y) - P_C(x)\|^2.
\]
Therefore, the operator \( P_C \) is firmly nonexpansive. From Cauchy inequality we conclude that
\[
\|P_C(x) - P_C(y)\| \leq \|x - y\|,
\]
that is, the operator \( P_C \) is nonexpansive.

Recall the notion of the subdifferential for an appropriate convex function.

\textbf{Definition 2.1 [12].} Let \( f : H \rightarrow \mathbb{R} \) be an appropriate convex. The subdifferential of \( f \) at \( x \) is defined as
\[
\partial f(x) = \{ \xi \in H \mid f(y) \geq f(x) + \langle \xi, y - x \rangle, \quad \forall y \in H \}.
\]
The lemma below is necessary for the convergence analysis in the next section.

\textbf{Lemma 2.1 [20].} Assume \( \varphi_k \in [0, \infty) \) and \( \delta_k \in [0, \infty) \) satisfy:
\begin{enumerate}
\item \( \varphi_{k+1} - \varphi_k \leq \theta_k (\varphi_k - \varphi_{k-1}) + \delta_k \),
\item \( \sum_{k=1}^{\infty} \delta_k < \infty \),
\item \( \{ \theta_k \} \subset [0, \theta] \), where \( \theta \in [0, 1) \).
\end{enumerate}
Then, the sequence \( \{ \varphi_k \} \) is convergent with \( \sum_{k=1}^{\infty} [\varphi_{k+1} - \varphi_k]_+ < \infty \), where \( [t]_+ := \max\{t, 0\} \) (for any \( t \in \mathbb{R} \)).

\textbf{Lemma 2.2 (Opial [21]).} Let \( H \) be a Hilbert space and \( \{ x^k \} \) a sequence such that there exists a nonempty set \( S \subset H \) verifying:
\begin{enumerate}
\item For every \( z \in S, \lim_{k \to \infty} \|x^k - z\| \) exists.
\item If \( x^{k_j} \rightharpoonup x^* \) weakly in \( H \) for a sequence \( k_j \to \infty \) then \( x \in S \).
\end{enumerate}
Then, there exists \( x \in S \) such that \( x^k \rightharpoonup x \) weakly in \( H \) as \( k \to \infty \).

3. The inertial relaxed- CQ algorithm and its asymptotic convergence.
3.1. The inertial relaxed-CQ algorithm. As in [24], we suppose the following conditions are satisfied:

1. The solution set of the SFP is nonempty.
2. The set \( C \) is denoted as 
   \[ C = \{ x \in H^1 \mid c(x) \leq 0 \}, \tag{3.1} \]
   where \( c : H^1 \to \mathbb{R} \) is appropriate convex and \( C \) is nonempty.

The set \( Q \) is denoted as 
\[ Q = \{ h \in H^2 \mid q(h) \leq 0 \}, \tag{3.2} \]
where \( q : H^2 \to \mathbb{R} \) is appropriate convex and \( Q \) is nonempty.

3. For any \( x \in H^1 \), at least one subgradient \( \xi \in \partial c(x) \) can be calculated.
   For any \( h \in H^2 \), at least one subgradient \( \eta \in \partial q(h) \) can be computed.

Now, we define two sets at point \( x_k \),
\[ C_k = \{ x \in H^1 \mid c(x_k) + \langle \xi_k, x - x_k \rangle \leq 0 \}, \tag{3.3} \]
where \( \xi_k \) is an element in \( \partial c(x_k) \),
\[ Q_k = \{ h \in H^2 \mid q(Ax_k) + \langle \eta_k, h - Ax_k \rangle \leq 0 \}, \tag{3.4} \]
where \( \eta_k \) is an element in \( \partial q(Ax_k) \).

By the definition of subgradient, it is clear that the halfspaces \( C_k \) and \( Q_k \) contain \( C \) and \( Q \), respectively. Due to the specific form of \( C_k \) and \( Q_k \), the orthogonal projections onto \( C_k \) and \( Q_k \) may be computed directly, see [4].

The following lemma provides an important boundedness property for the subdifferential.

**Lemma 3.1 [12].** Suppose that \( f : H \to \mathbb{R} \) is convex. Then its subdifferential are uniformly bounded on any bounded subsets of finite dimensional dimensional space \( H \).

Next, we state our inertial relaxed-CQ algorithm.

**Algorithm 3.1**

Initialization: Take \( x^0, x^1 \) in \( H^1 \).

Iterative step: For \( k \geq 0 \), given the points \( x^k, x^{k-1} \), the next iterative point \( x^{k+1} \) is generated by 
\[ x^{k+1} = P_{C_k}[U_k(x^k + \theta_k(x^k - x^{k-1}))], \tag{3.5} \]
where \( U_k = I - \gamma F_k, F_k = A^T(I - P_{Q_k})A, \gamma \in (0, 2/L), L \) denotes the spectral radius of \( A^T A, \theta_k \in [0, 1], C_k \) and \( Q_k \) are given by (3.3) and (3.4), respectively.

Evidently, when \( \theta_k \equiv 0 \), (3.5) happens to be the standard relaxed CQ method.

3.2. Asymptotic convergence of the inertial relaxed-CQ algorithm. In this subsection, we establish the asymptotic convergence of Algorithm 3.1.

Define \( f(x) = \frac{1}{2}\|Ax - P_{Q}Ax\|^2, x \in H^1 \). Assuming that the SFP(1.1) is consistent, it is not hard to see that \( x \) solves the SFP (1.1) if and only if \( x \) solves the minimization \( f_{\min} := \min_{x \in C} f(x) \) with \( f_{\min} = 0 \). It is well known that the gradient of \( f : F = \nabla f = A^T(I - P_{Q})A \) is \( L \)-Lipschitz continuous with \( L = \rho(A^T A) \), and
Thus it is $\frac{1}{L}$-cocoercive (see [6]), that is $\langle F(x) - F(y), x - y \rangle \geq \frac{1}{L} \|F(x) - F(y)\|^2$. The same is true for the operators $F_k = AT(I - P_{q_k})A$ for $k = 0, 1, \ldots$.

**Theorem 3.1.** Suppose (1.1) is consistent. Choose parameter $\theta_k \in [0, \tilde{\theta}_k]$ with $
\tilde{\theta}_k := \min\{\theta, \max\{k/\|x^k - x^{k-1}\| + 1/2, k/\|x^k - x^{k-1}\| + 1\}\}$, $\theta \in [0, 1)$, then the sequence $\{x^k\}$ generated by (3.5) converges weakly to a point $x^*$ as $k \to \infty$, where $x^*$ is a solution of (1.1).

**Proof.** The case for $\theta_k \equiv 0$, we can see the detailed proof in [23].

Now we see the case for $\theta_k > 0$ for some $k \in N$. Let $z$ be a solution of the SFP. Since $C \subseteq C_k, Q \subseteq Q_k$, then $z = P_C(z) = P_{C_k}(z)$ and $F_k(z) = F(z) = 0$. Define the auxiliary real sequence $\varphi_k := \frac{1}{2}\|x^k - z\|^2$. Let $y^k = x^k + \theta_k(x^k - x^{k-1})$. Therefore, from the nonexpansive of the operator $P_{C}$ and (3.5), we have

\[
\varphi_{k+1} = \frac{1}{2}\|x^{k+1} - z\|^2 \\
= \frac{1}{2}\|P_{C_k}[U_k(x^k + \theta_k(x^k - x^{k-1}))] - z\|^2 \\
\leq \frac{1}{2}\|y^k - z\|^2 - \gamma F_k(y^k) - F_k(z) \\
= \frac{1}{2}\|y^k - z\|^2 + \frac{\gamma^2}{2}\|F_k(y^k)\|^2 - \gamma\langle y^k - z, F_k(y^k) - F_k(z)\rangle.
\]

Furthermore, by the cocoercivity of the operator $F_k$, we have

\[
\varphi_{k+1} \leq \frac{1}{2}\|y^k - z\|^2 + \frac{\gamma^2}{2}\|F_k(y^k)\|^2 - \gamma\frac{1}{L}\|F_k(y^k)\|^2
\]

namely,

\[
\varphi_{k+1} \leq \frac{1}{2}\|y^k - z\|^2 - (\frac{\gamma}{L} - \frac{\gamma^2}{2})\|A^T(P_{Q_k} - I)Ay^k\|^2.
\]

By deducing, we have

\[
\frac{1}{2}\|y^k - z\|^2 = \frac{1}{2}\|x^k + \theta_k(x^k - x^{k-1}) - z\|^2 \\
\leq \frac{1}{2}\|x^k - z\|^2 + \theta_k\langle x^k - z, x^k - x^{k-1}\rangle + \frac{\theta_k^2}{2}\|x^k - x^{k-1}\|^2 \\
= \varphi_k + \theta_k\langle x^k - z, x^k - x^{k-1}\rangle + \frac{\theta_k^2}{2}\|x^k - x^{k-1}\|^2.
\]

It is easy to check that $\varphi_k = \varphi_{k-1} + \langle x^k - z, x^k - x^{k-1}\rangle - \frac{1}{2}\|x^k - x^{k-1}\|^2$. Hence

\[
\frac{1}{2}\|y^k - z\|^2 = \varphi_k + \theta_k\langle \varphi_k - \varphi_{k-1}\rangle + \frac{\theta_k + \theta_k^2}{2}\|x^k - x^{k-1}\|^2.
\]

Putting (3.7) into (3.6), we get

\[
\varphi_{k+1} \leq \varphi_k + \theta_k\langle \varphi_k - \varphi_{k-1}\rangle + \frac{\theta_k + \theta_k^2}{2}\|x^k - x^{k-1}\|^2 - (\frac{\gamma}{L} - \frac{\gamma^2}{2})\|A^T(P_{Q_k} - I)Ay^k\|^2.
\]

Since $0 < \gamma < 2/L$, we have $(\frac{\gamma}{L} - \frac{\gamma^2}{2}) > 0$. According to $\theta_k^2 \leq \theta_k$ and (3.8), we derive

\[
\varphi_{k+1} \leq \varphi_k + \theta_k\langle \varphi_k - \varphi_{k-1}\rangle + \theta_k\|x^k - x^{k-1}\|^2.
\]
Evidently,
\[ \sum_{k=1}^{+\infty} \theta_k \| x^k - x^{k-1} \|^2 < \infty, \quad (3.10) \]
due to \( \theta_k \| x^k - x^{k-1} \|^2 \leq \frac{1}{L^2} \). Let \( \delta_k := \theta_k \| x^k - x^{k-1} \|^2 \) in Lemma 2.1. We deduce that the sequence \( \{ \| x^k - z \| \} \) is convergent (hence \( \{ x^k \} \) is bounded). By (3.9) and Lemma 2.1, we obtain \( \sum_{k=1}^{+\infty} \| x^k - z \|^2 - \| x^{k-1} - z \|^2 \| < \infty \). By reason of (3.6), we have
\[ \left( \frac{\gamma}{L} - \frac{\gamma^2}{2} \right) \| A^T (P_{Q_k} - I) A y^k \|^2 \leq \varphi_k - \varphi_{k+1} + \theta_k (\varphi_k - \varphi_{k-1}) + \theta_k \| x^k - x^{k-1} \|^2. \]
Therefore
\[ \sum_{k=1}^{+\infty} \left( \frac{\gamma}{L} - \frac{\gamma^2}{2} \right) \| A^T (P_{Q_k} - I) A y^k \|^2 < \infty. \]
By \( 0 < \gamma < 2/L \), we get
\[ \| A^T (P_{Q_k} - I) A y^k \|^2 \to 0 \quad (3.11) \]
and
\[ \| (P_{Q_k} - I) A y^k \|^2 \to 0. \quad (3.12) \]
Obviously, \( F_k(y^k) \to 0 \). We next show that
\[ \| x^{k+1} - x^k \| \to 0. \quad (3.13) \]
To do this, we proceed as follows:
\[
\| x^{k+1} - z \|^2 = \| (x^{k+1} - y^k) + (y^k - z) \|^2 \\
= \| x^{k+1} - y^k \|^2 + \| y^k - z \|^2 + 2(x^{k+1} - y^k, y^k - z) \\
= \| x^{k+1} - y^k \|^2 + \| y^k - z \|^2 \\
+ 2(x^{k+1} - y^k, y^k - x^{k+1}) + 2(x^{k+1} - y^k, x^{k+1} - z).
\]
Hence
\[ \| x^{k+1} - y^k \|^2 = \| y^k - z \|^2 - \| x^{k+1} - z \|^2 + 2(x^{k+1} - y^k, x^{k+1} - z). \quad (3.14) \]
Putting (3.7) into (3.14), we have
\[
\| x^{k+1} - y^k \|^2 = 2\varphi_k + 2\theta_k (\varphi_k - \varphi_{k-1}) - \| x^{k+1} - z \|^2 + (\theta_k + \theta_k^2) \| x^k - x^{k-1} \|^2 \\
+ 2(x^{k+1} - y^k, x^{k+1} - z) \\
= \| x^k - z \|^2 - \| x^{k+1} - z \|^2 + \theta_k (\| x^k - z \|^2 - \| x^{k+1} - z \|^2) \\
+ (\theta_k + \theta_k^2) \| x^k - x^{k-1} \|^2 + 2(x^{k+1} - y^k, x^{k+1} - z).
\]
By \( \theta_k \in [0, \bar{\theta}_k] \), \( \bar{\theta}_k := \min\{ \theta, \frac{1}{\max\{\theta_k\| x^k - x^{k-1} \|^2, \theta_k\| x^k - x^{k-1} \|^2\}} \} \), \( \theta \in [0, 1) \), we get
\[
\| x^{k+1} - y^k \|^2 \leq \left( \| x^k - z \|^2 - \| x^{k+1} - z \|^2 \right) + \theta_k (\| x^k - z \|^2 - \| x^{k+1} - z \|^2) + 2\theta_k \| x^k - x^{k-1} \|^2 \\
+ 2(x^{k+1} - y^k, x^{k+1} - z), \quad (3.15)
\]
where \( t_+ = \max\{ t, 0 \} \). On the other hand, \( x^{k+1} = P_{C_k} (x^k - \gamma F_k(y^k)) \) implies
\[ \langle (y^k - \gamma F_k(y^k)) - x^{k+1}, z - x^{k+1} \rangle \leq 0. \]
Then,
\[ \langle x^{k+1} - y^k, x^{k+1} - z \rangle \leq \gamma \langle F_k(y^k), z - x^{k+1} \rangle \to 0. \quad (3.16) \]
From (3.10), we obtain
\[ \lim_{k \to 1} \theta_k \| x^k - x^{k-1} \|^2 = 0 \quad (3.17) \]
Combining (3.15) with (3.16) and (3.17) leads to
\[ \|x^{k+1} - y^k\| \to 0. \]  
(3.18)

By the triangle inequality, we get
\[ \|x^{k+1} - x^k\| \leq \|x^{k+1} - y^k\| + \|y^k - x^k\| = \|x^{k+1} - y^k\| + \theta_k\|x^k - x^{k-1}\|. \]  
(3.19)

Since \( \theta_k \in [0, \tilde{\theta}_k] \) with \( \tilde{\theta}_k := \min\{\theta, \frac{1}{\max\{\aleph^2\|x^k - x^{k-1}\|, \aleph\|x^k - x^{k-1}\|\}}\}, \theta \in [0, 1) \), we have
\[ \sum_{k=1}^{+\infty} \theta_k\|x^k - x^{k-1}\| < \infty, \]  
(3.20)

this implies
\[ \theta_k\|x^k - x^{k-1}\| \to 0. \]  
(3.21)

Using (3.18) and (3.21), from (3.19), we derive that
\[ \|x^{k+1} - x^k\| \to 0. \]

We have known that \( \{x^k\} \) is bounded, which implies that \( \xi_k \) is bounded. Assume that \( x^* \) is an accumulation point of \( \{x^k\} \) and \( x^{k_l} \to x^* \), where \( \{x^{k_l}\} \) is a subsequence of \( \{x^k\} \). Since \( \theta_k\|x^k - x^{k-1}\| \to 0 \), we have \( y^{k_l} \to x^* \). Then, from (3.11), it follows
\[ P_{Q_{k_l}}(A_{x^{k_l}}) \to A_{x^*}, \quad k_l \to +\infty. \]  
(3.22)

Finally, we show that \( x^* \) is a solution of the SFP. Since \( x^{k_l+1} \in C_{k_l} \), it is obtained that
\[ c(x^{k_l}) + \langle \xi^{k_l}, x^{k_l+1} - x^{k_l} \rangle \leq 0. \]

Thus
\[ c(x^{k_l}) \leq -\langle \xi^{k_l}, x^{k_l+1} - x^{k_l} \rangle \leq \xi\|x^{k_l+1} - x^{k_l}\|, \]

where \( \xi \) satisfies \( \|\xi^k\| \leq \xi \) for all \( k \). By virtue of the continuity of function \( c \) and \( \|x^{k_l+1} - x^k\| \to 0 \), we get that
\[ c(x^*) = \lim_{l \to \infty} c(x^{k_l}) \leq 0. \]

Therefore, \( x^* \in C \).

Now we show that \( Ax^* \in Q \). To do this, let \( z^k = Ay^k - P_{Q_{k_l}}(Ay^k) \to 0 \) and let \( \eta \) be such that \( \|\eta_k\| \leq \eta \) for all \( k \). Since \( Ay^{k_l} - z^{k_l} = P_{Q_{k_l}}(Ay^{k_l}) \in Q_{k_l} \), we have
\[ q(Ax^{k_l}) + \langle \eta^{k_l}, (Ay^{k_l} - z^{k_l}) - Ax^{k_l} \rangle \leq 0. \]

Hence,
\[ q(Ax^{k_l}) \leq \langle \eta^{k_l}, Ax^{k_l} - Ay^{k_l} \rangle + \langle \eta^{k_l}, z^{k_l} \rangle \leq \eta\|Ax^{k_l} - x^{k_l-1}\| + \eta\|z^{k_l}\| \to 0. \]

By the continuity of \( q \) and \( A(x^{k_l}) \to Ax^* \), we arrive at the conclusion
\[ q(Ax^*) = \lim_{l \to \infty} q(Ax^{k_l}) \leq 0, \]

namely \( Ax^* \in Q \).

Hence, by Lemma 2.2, the result of this theorem can be obtained.

**Remark 3.1.** Since the current value of \( \|x^k - x^{k-1}\| \) is known before choosing the parameter \( \theta_k \), then \( \theta_k \) is well-defined in Theorem 3.1. In fact, from the process of proof for the theorem 3.1, we can get the following assert: The convergence result...
of Theorem 3.1 always holds provided that we take \( \theta_k \in [0, \theta], \) \( \theta \in [0, 1), \forall k \geq 0, \) with
\[
\sum_{k=1}^{+\infty} \theta_k \|x^k - x^{k-1}\|^2 < \infty
\]
and
\[
\sum_{k=1}^{+\infty} \theta_k \|x^k - x^{k-1}\| < \infty.
\]

4. A modified inertial relaxed-CQ algorithm and its asymptotic convergence. In this section, a modified inertial relaxed-CQ algorithm is presented, the asymptotic convergence is shown under some conditions.

**Algorithm 4.1.** Take \( x^0, x^1 \in R^N \), the sequence \( \{x^k\}_{k \geq 0} \) is generated by the iterative process
\[
x^{k+1} = (1 - \alpha_k)y^k + \alpha_k P_{C_k}[U_k(y^k)],
\]
where \( U_k = I - \gamma F_k, F_k = A^T(I - P_{Q_k})A \), \( y^k = x^k + \theta_k(x^k - x^{k-1}) \), \( \alpha_k \in (0, 1), \theta_k \in [0, 1), \gamma, C_k, Q_k \) are given as in Algorithm 3.1.

Now, we establish the asymptotic convergence of the algorithm 4.1.

**Theorem 4.1.** Suppose (1.1) is consistent, \( \theta_k \in [0, \theta], \theta \in [0, 1), \forall k \geq 0. \) If (3.12), (3.21) and the following condition holds
\[
1 > R_1 = \inf_{k \geq 0} \alpha_k > 0,
\]
then the sequence \( \{x^k\} \) generated by (4.1) converges weakly to a point \( x^* \) contained in the set of solution of (1.1).

**Proof.** Let \( z \) be a solution of the SFP. \( C \subset C_k \) implies \( z = P_C(z) = P_{C_k}(z) \). Define the auxiliary real sequence \( \varphi_k := \frac{1}{2} \|x^k - z\|^2 \). From (4.1), we have
\[
\varphi_{k+1} = \frac{1}{2} \|x^{k+1} - z\|^2
\]
\[
= \frac{1}{2} \|(1 - \alpha_k)y^k + \alpha_k P_{C_k}U_k(x^k + \theta_k(x^k - x^{k-1})) - z\|^2
\]
\[
= \frac{1}{2} \|(1 - \alpha_k)y^k + \alpha_k P_{C_k}[y^k - \gamma F_k(y^k)] - z\|^2
\]
\[
\leq \frac{1}{2} (1 - \alpha_k) \|y^k - z\|^2 + \frac{1}{2} \alpha_k \|P_{C_k}[y^k - \gamma F_k(y^k)] - PC_k z\|^2
\]
\[
\leq \frac{1}{2} (1 - \alpha_k) \|y^k - z\|^2 + \frac{1}{2} \alpha_k \|y^k - \gamma F_k(y^k) - z\|^2
\]
\[
= \frac{1}{2} (1 - \alpha_k) \|y^k - z\|^2 + \frac{1}{2} \alpha_k \|y^k - z\|^2 + \alpha_k \frac{\gamma^2}{2} \|F_k(y^k)\|^2
\]
\[
- \alpha_k \gamma \langle y^k - z, F_k(y^k) - F_k(z) \rangle.
\]
So we get
\[
\varphi_{k+1} \leq \frac{1}{2} \|y^k - z\|^2 + \alpha_k \frac{\gamma^2}{2} \|F_k(y^k)\|^2 - \alpha_k \frac{\gamma}{L} \|F_k(y^k)\|^2,
\]
therefore
\[
\varphi_{k+1} \leq \frac{1}{2} \|y^k - z\|^2 - \alpha_k \left( \frac{\gamma}{L} - \frac{\gamma^2}{2} \right) \|F_k(y^k)\|^2.
\]
Putting (3.10) into (4.4), we get
\[ \varphi_{k+1} \leq \varphi_k + \theta_k(\varphi_k - \varphi_{k-1}) + \theta_k\|x^k - x^{k-1}\|^2 - \alpha_k\left(\frac{\gamma}{L} - \frac{\gamma^2}{2}\right)\|A^T(P_k - I)Ay^k\|^2. \] (4.5)
Since \(0 < \gamma < 2/L\) and \(0 < \inf \alpha_k < 1\), we have
\[ \varphi_{k+1} \leq \varphi_k + \theta_k(\varphi_k - \varphi_{k-1}) + \theta_k\|x^k - x^{k-1}\|^2 - R_1\left(\frac{\gamma}{L} - \frac{\gamma^2}{2}\right)\|A^T(P_k - I)Ay^k\|^2, \] (4.6)
moreover
\[ \varphi_{k+1} \leq \varphi_k + \theta_k(\varphi_k - \varphi_{k-1}) + \theta_k\|x^k - x^{k-1}\|^2. \] (4.7)
Suppose \(\sum_{k=1}^{+\infty} \theta_k\|x^k - x^{k-1}\|^2 < \infty\), choose \(\delta_k := \theta_k\|x^k - x^{k-1}\|^2\) in Lemma 2.1, we deduce that the sequence \(\{\|x^k - z\|\}\) is convergent (hence \(\{x^k\}\) is bounded). From (4.7) and Lemma 2.1 we obtain \(\sum_{k=1}^{+\infty} [\|x^k - z\|^2 - \|x^{k-1} - z\|^2]^+ < \infty\), while from (4.6) we have
\[ R_1\left(\frac{\gamma}{L} - \frac{\gamma^2}{2}\right)\|A^T(P_k - I)Ay^k\|^2 \leq \varphi_k - \varphi_{k+1} + \theta_k(\varphi_k - \varphi_{k-1}) + \theta_k\|x^k - x^{k-1}\|^2. \]
Obviously,
\[ \sum_{k=1}^{+\infty} R_1\left(\frac{\gamma}{L} - \frac{\gamma^2}{2}\right)\|A^T(P_k - I)Ay^k\|^2 < \infty. \]
Since \(0 < \gamma < 2/L\) and \(0 < R_1 < 1\), we have
\[ \|A^T(P_k - I)Ay^k\|^2 \rightarrow 0. \] (4.8)
The rest part of the proof is similar to that of Theorem 3.1, and hence it is omitted.

5. Numerical results. In this section, we will test three numerical experiments.
Throughout the computational experiments, we set \(\varepsilon = 10^{-4}\). In the algorithms, we take \(\gamma = 1/L\), \(L\) denotes the spectral radius of \(A^TA\), \(\theta = 0.8\). If \(\theta \leq \frac{1}{\max\{k\|x^k - x^{k-1}\|^2, k\|x^k - x^{k-1}\|\}}\), we take \(\theta_k = \frac{\theta}{2} = 0.4\); otherwise, we take \(\theta_k = \frac{1}{\max\{(k+1)^2\|x^k - x^{k-1}\|^2, (k+1)^2\|x^k - x^{k-1}\|\}}\), \(k = 1, 2, \ldots\).

Example 5.1 Let
\[ A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
\[ C = \{x \in \mathbb{R}^3| x_1^2 + x_2^2 - 9 \leq 0\}; \]
\[ Q = \{x \in \mathbb{R}^3| x_1 + x_3 - 3 \leq 0\}. \]
Find \(x \in C\) with \(Ax \in Q\).

Example 5.2 Let
\[ A = \begin{bmatrix} 2 & -1 & 3 & 2 & 3 \\ 1 & 2 & 5 & 2 & 1 \\ 2 & 0 & 2 & 1 & -2 \\ 2 & -1 & 0 & -3 & 5 \end{bmatrix} \]
\[ C = \{x \in \mathbb{R}^3| x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 0.25 \leq 0\}; \]
\[ Q = \{x \in \mathbb{R}^4| x_1 + x_2 + x_3 + x_4 + 0.6 \leq 0\}. \]
Find \(x \in C\) with \(Ax \in Q\).

Example 5.3 Let \(A = (a_{ij})_{M \times N}, a_{ij} \in (0, 1)\) be a random matrix, \(M, N\) are two positive integers. \(C = \{x \in \mathbb{R}^N| \sum_{i=1}^{N} x_i^2 \leq r^2\}; \)
\[ Q = \{x \in \mathbb{R}^M| x \leq b\}. \]
To ensure the existence of the solution of the problem, the vector \(b\) is generated by using the
Table 1. The numerical results of example 5.1

<table>
<thead>
<tr>
<th>Initiative point</th>
<th>R-Iter</th>
<th>Iner-R-Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^0 = (3.2, 4.2, 5.2)$</td>
<td>$k = 74; s = 0.068$</td>
<td>$k = 5; s = 0.016$</td>
</tr>
<tr>
<td>$x^1 = (-0.5843, 2.3078, 3.3435)$</td>
<td>$x^* = (-0.6200, 1.6180, 1.6216)$</td>
<td>$x^* = (-1.1281, 1.0720, 1.9694)$</td>
</tr>
<tr>
<td>$x^0 = (10, 0, 10)$</td>
<td>$k = 93; s = 0.090$</td>
<td>$k = 84; s = 0.085$</td>
</tr>
<tr>
<td>$x^1 = (2.0825, -2.5275, 6.4589)$</td>
<td>$x^* = (0.9000, -1.7152, 1.7074)$</td>
<td>$x^* = (-0.1061, -1.4514, 2.1596)$</td>
</tr>
<tr>
<td>$x^0 = (2, -5, 2)$</td>
<td>$k = 73; s = 0.075$</td>
<td>$k = 35; s = 0.035$</td>
</tr>
<tr>
<td>$x^1 = (1.3327, -3.2657, 1.9328)$</td>
<td>$x^* = (1.1512, -2.7679, 1.8616)$</td>
<td>$x^* = (0.9010, -2.1029, 1.8169)$</td>
</tr>
</tbody>
</table>

Table 2. The numerical results of example 5.1

<table>
<thead>
<tr>
<th>Initiative point</th>
<th>$\alpha_k$</th>
<th>Iner-KM-R-Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^0 = (3.2, 4.2, 5.2)$</td>
<td>0.4</td>
<td>$k = 3; s = 0.016$</td>
</tr>
<tr>
<td>$x^1 = (-0.5843, 2.3078, 3.3435)$</td>
<td>0.8</td>
<td>$k = 3; s = 0.013$</td>
</tr>
<tr>
<td>$x^* = (-2.6931, 1.2534, 2.2937)$</td>
<td></td>
<td>$x^* = (-2.6828, 1.2585, 2.2835)$</td>
</tr>
<tr>
<td>$x^0 = (10, 0, 10)$</td>
<td>0.4</td>
<td>$k = 76; s = 0.086$</td>
</tr>
<tr>
<td>$x^1 = (2.0825, -2.5275, 6.4589)$</td>
<td>0.8</td>
<td>$k = 74; s = 0.085$</td>
</tr>
<tr>
<td>$x^* = (-0.1346, -2.6392, 2.3046)$</td>
<td></td>
<td>$x^* = (-0.0799, -2.6190, 2.3611)$</td>
</tr>
<tr>
<td>$x^0 = (2, -5, 2)$</td>
<td>0.6</td>
<td>$k = 62; s = 0.056$</td>
</tr>
<tr>
<td>$x^1 = (1.3327, -3.2657, 1.9328)$</td>
<td>0.8</td>
<td>$k = 45; s = 0.046$</td>
</tr>
<tr>
<td>$x^* = (0.9006, -2.1031, 1.8171)$</td>
<td></td>
<td>$x^* = (0.9008, -2.1030, 1.8170)$</td>
</tr>
</tbody>
</table>

Table 3. The numerical results of example 5.2

<table>
<thead>
<tr>
<th>Initiative point</th>
<th>R-Iter</th>
<th>Iner-R-Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^0 = (0, 0, 0, 0)$</td>
<td>$k = 15; s = 0.675$</td>
<td>$k = 5; s = 0.018$</td>
</tr>
<tr>
<td>$x^1 = (-0.0092, 0, -0.0132, -0.0026, -0.0092)$</td>
<td>$x^* = (-0.0208, 0, -0.0297, -0.0059, -0.0020)$</td>
<td>$x^* = (0.0015, 0, -0.0412, -0.0082, -0.0088)$</td>
</tr>
<tr>
<td>$x^0 = (1, 1, 1, 1)$</td>
<td>$k = 20; s = 0.083$</td>
<td>$k = 3; s = 0.0272$</td>
</tr>
<tr>
<td>$x^1 = (0.3237, 0.5471, 0.2280, 0.4833, 0.3237)$</td>
<td>$x^* = (0.0171, 0.3822, 0.1394, 0.2779, 0.0171)$</td>
<td>$x^* = (-0.0784, 0.2935, -0.2378, 0.1873, -0.0784)$</td>
</tr>
<tr>
<td>$x^0 = (20, 10, 20, 10, 20)$</td>
<td>$k = 22; s = 0.090$</td>
<td>$k = 6; s = 0.067$</td>
</tr>
<tr>
<td>$x^1 = (6.1605, 5.0023, 4.5130, 3.9040, 6.1605)$</td>
<td>$x^* = (0.837, 0.3910, -0.2155, 0.1915, 0.0837)$</td>
<td>$x^* = (-0.2490, -0.2117, -0.1742, -0.1619, -0.2490)$</td>
</tr>
</tbody>
</table>
The numerical results of Example 5.2 with

\[ x^0 = (0, 0, 0, 0) \]
\[ x^1 = (-0.0009, 0, -0.0132, 0.0026) \]
\[ \alpha = 0.6 \]
\[ k = 6; s = 0.020 \]

\[ x^0 = (1, 1, 1, 1) \]
\[ x^1 = (0.3237, 0.5471, 0.2280, 0.4833, 0.3237) \]
\[ \alpha = 0.4 \]
\[ k = 3; s = 0.034 \]

\[ x^0 = (20, 10, 20, 10, 20) \]
\[ x^1 = (6.1605, 5.0023, 4.5130, 3.9040, 6.1605) \]
\[ \alpha = 0.8 \]
\[ k = 7; s = 0.071 \]

The numerical results of Example 5.3 with

\[ x^0 \in C, r = ||x^*||, \text{ taking } b = Ax^*. \]
\[ x \in C \text{ with } Ax \in Q. \]

The numerical results of Examples 5.1-5.3 can
be seen from Tables 1-5. In Tables 1-5, "R-Iter", "Iner-R-Iter" and "Iner-KM-R-Iter" denote the relaxed CQ algorithm, the inertial relaxed CQ algorithm and the modified inertial relaxed CQ algorithm, respectively. "k" and "s" denote the number of iterations, cpu time in seconds and the solution, respectively. To compare conveniently, we take the initial point \( x^1 \) in the latter two algorithms as in the R-iter, that is, the point \( x^1 \) is generated by the R-iter.

Table 1 and Table 2 give the numerical results of Example 5.1 with the R-Iter, the Iner-R-Iter, and the Iner-KM-R-Iter for different \( \alpha_k \), respectively. Table 3 and Table 4 show the numerical results of Example 5.2 with the R-Iter, the Iner-R-Iter, and the Iner-KM-R-Iter for different \( \alpha_k \), respectively. Table 5 gives the numerical results of Example 5.3 with \( \alpha_k = 0.6, k = 1, 2, \ldots \).

From Tables 1-5, we can see that our algorithms are effective and they converge more quickly than the relaxed-CQ algorithm in [23]. Notice that the greater value of \( \alpha_k \), the faster the convergence of the Iner-KM-R-Iter.

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